Using Error Analysis to Teach Equation Solving

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IN MANY MATHEMATICS CLASSROOMS, STUdents are responsible for correcting their own homework. Placing a red X next to an incorrect answer is often the extent of homework checking because students often lack the ability to find and correct mistakes in their own work. Without this skill, students lose a valuable opportunity to discuss homework in a meaningful way. When classroom activities and discussion focus on analyzing errors, students realize that finding a numeric answer is an insufficient goal and become more willing to show and explain their work. Homework discussions begin to focus on "How did you get that?" rather than being limited to "Is my answer right?"

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According to the NCTM's *Principles and Standards for School Mathematics*, "Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically" (NCTM 2000, p. 60). This article presents several Standards-based activities and problems that use error analysis to improve students' ability to read and discuss mathematics work. In the first two activities, students are required to explain their thinking and listen carefully to the thinking of other students.

Written on the Board	Student Explanation
$2(x + 1) = x - 4(x - 2)$	Teacher presents problem.
$2x + 2 = x - 4(x - 2)$	Student 1 uses the distributive property.
$2x + 2 = x - 4x - 8$	Student 2 uses the distributive property incorrectly. This step is later circled and identified as a common error.
$2x + 2 = x + -4(x + -2)$	Amanda writes her idea on the board.
$2x + 2 = x + -4x + 8$	Student 2 corrects the error after discussion.
$2x + 2 = -3x + 8$	Student 3 combines like terms.
$3x + 2x + 2 = 3x + -3x + 8$ $5x + 2 = 8$	Student 4 adds $3x$ to both sides to get all the variable terms on one side.
$5x + 2 - 2 = 8 - 2$ $5x = 6$	Student 5 writes an alternate solution from his homework paper, explaining that he subtracted 2. $2x + 2 - 2 = -3x + 8 - 2$ $2x = -3x + 6$ Then he subtracts 2 from student 4's solution.
$x = 6/5$	Student 6 divides both sides by 5 because division "undoes" multiplication.

Fig. 1 A typical problem solved during "pass the pen"

Through the third activity, in-class journal writing and take-home writing projects, students develop an understanding of common errors and improve their ability to communicate effectively with others when correcting homework. Although all the examples come from a unit on linear equation solving, the ideas and activities presented in this article have been used successfully in other algebra units and in other courses.

Activity 1: Pass the Pen

ACCORDING TO THE NCTM'S STANDARDS, "FOR some students, participation in class discussions is a challenge" (NCTM 2000, p. 61). The "pass the pen" activity is structured so that class discussion involves every student. The teacher writes a multistep mathematics problem on the whiteboard and calls on one student to come up and complete the first step. The teacher gives the student a whiteboard marker, or pen, to write his or her work. The student explains not only how to complete the step but also the reasoning behind it. The student holding the pen then calls on a second student to complete the next step, and "passes the pen." After completing a step, each student passes the pen to another until the problem is finished. When a question arises, the person holding the pen must answer the question, call on another student for help, or pass the pen to a different student. Typically during this activity, one or more errors will be made by students, as in the following scenario referring to **figure 1**.

Student 2: I used the distributive property, like in the first step. [*Hands go up all over the room. Student 2 is alerted by this action that she has made a mistake and looks to the teacher.*] Isn't that right?

Teacher: If you have a question, why don't you call on someone to help you? [*Student 2 looks around the room and chooses Mario.*]

Mario: You forgot about the subtraction sign in front of the 4.

Student 2: [*Obviously still confused.*] I don't get it. That's why it's subtracting $4x$. I multiplied the 4 and the 2 to get 8. [*She calls on Amanda for more assistance.*]

Amanda: If you write subtraction as adding the opposite, the problem is –4 times –2 and that's +8. You got –8 because you didn't look at both negatives.

Teacher: Amanda, will you write your idea on the board so that everyone can better understand what you are saying? [*Student 2 hands Amanda the pen, and she writes her idea on the board. When Amanda is finished, she returns the pen to student 2.*]

Student 2: Oh, I get it. A negative times a negative is a positive. You have to write the subtraction as an addition problem to see both negatives. [*She corrects her work and passes the pen to a new student.*]

Solving a problem in this manner will clearly take more time than a teacher-presented solution given during a lecture. The extra time is necessary, because students require time to think between each step and formulate their questions. During a lecture, many middle-level students have difficulty simultaneously

copying notes and processing the ideas. As an additional benefit, the teacher has an informal opportunity to assess what concepts are still giving students difficulty, as in the dialogue below when student 5 is uncertain how to proceed.

Student 5: That's not what I got [*referring to student 4's work*]. I don't know what to do next!

Teacher: Why don't you write what you did on the board and call on someone to help you? [*Student 5 copies work from his homework paper.*]

Student 5: [*Looking for hands.*] I got $2x = -3x + 6$; does that mean I'm wrong? [*Student 5 calls on Tran.*]

Tran: I did what you did, and I think it's right because I just subtracted 2 from both sides then added $3x$. Student 3 has already added the $3x$, so you can subtract 2 from both sides now, and it will still work. [*Student 5 subtracts 2 from both sides.*]

Teacher: [*To student 5.*] Does that agree with the answer you got on your homework?

Student 5: Yes. [*Student pauses and thinks.*] Can you always do the steps in any order?

The discussion continues by students passing the pen to other students. When disagreements or different solutions occur during the discussion, it is helpful to write alternative solutions on the board in addition to the original solution. Students will benefit from the opportunity to compare solutions and judge whether the differing solutions are both correct. If both solutions are correct, students can form their own opinion about which solution is easier. If a solution is incorrect, it is helpful to add notes to clarify the error so that students can avoid a similar mistake in future work.

A key element of "pass the pen" is that everyone must participate. No one can go to the board a second time until everyone has held the pen at least once. Students who usually do not present solutions at the board know that they will have to take a turn and look for an opportunity to participate. This activity is also effective because the person who holds the pen is always in charge of the discussion, not the teacher. Thus, although everyone must take a turn, he or she can always call on another student for help or pass the pen if feeling uncomfortable. The role of the teacher during "pass the pen" is to enforce the participation rules and ask questions that will take the discussion deeper: Why did you do that? Is that the only possible next step? Will that always work? How is that different from the last problem?

Activity 2: Find an Expert

IN THIS ACTIVITY, STUDENTS TAKE TURNS BEING the expert and the questioner. It is a useful way for students to correct their own work on a quiz or homework assignment. Students love the opportunity to move around the room and work with different people. They also appreciate playing different roles in the activity. The teacher can assign roles for particular problems or allow students to select their own partners.

Students are instructed to write in a binder the correct solutions to the problems they missed by consulting a different expert for each missed problem. An expert, someone who solved the problem correctly, is instructed not to show his or her own paper to anyone but to find mistakes and answer questions. The questioner, the student who missed the problem, is to ask enough questions to be able to write the correct solution on the binder paper.

"It is difficult for students to learn to consider, evaluate, and build on the thinking of others, especially when their peers are still developing their own mathematical understandings" (NCTM 2000, p. 63). The rules of this activity discourage copying and encourage discussion of students' thinking. Because the expert cannot write on the binder paper, he or she must use words to explain his or her thinking. It is useful to model good questions before the activity begins. For example, "Why did you divide by 3?" is a more thoughtful question than "What should I write next?" During the activity, the teacher monitors the

Common Error	Reason for Confusion (False Analogy)
$2x + 3x = 5x^2$	Students confuse $2x + 3x = 5x$, where only the coefficients are added, with $2x \cdot 3x = 6x^2$, where the coefficients as well as the variable are multiplied.
$-3^2 = 9$	Students remember that the square of a negative is positive but forget the importance of parentheses. $(-3)^2 = +9$ and $-3^2 = -9$
$6x - (x - 3) = 6x - x - 3$	Students fail to understand that subtracting quantities is different from adding quantities. Another problem can look similar to students. $6x + (3x - 3) = 6x + 3x - 3$
$-\|-2\| = 2$	Students find it difficult to distinguish between absolute value and parentheses. $-(-2) = +2$ but $-\|-2\| = -2$

Fig. 2 Various ideas for writing prompts

discussions in the classroom. A conversation where an expert explains a solution incorrectly provides an excellent opportunity for the teacher to ask clarifying questions of his or her own. Sometimes a pair of students become stumped and are unable to figure out the mistake in the questioner's method. This situation also provides an excellent problem for further class discussion or journal writing.

Activity 3: Journal Writing

STUDENTS ARE PREPARED TO DESCRIBE THEIR thinking in journals after discussing problem solutions and working collaboratively to find errors. It is helpful to have students write in journals on a regular basis and organize journal entries around common errors (see **fig. 2**). A careful reading of student responses can give insight into the current depth of understanding for each student. When asked to find the mistake in the following journal entry, three students demonstrate different levels of understanding (see **fig. 3**).

Gabrielle attempts to prove the simplification is incorrect by evaluating the expression. Although she is clearly aware that the order of operations is required, her work is unclear and she is unable to find the mistake or the correct simplification. Prabdheep is also unable to correctly simplify the expression, although his explanation of the distributive property is clear. He correctly explains that the +6 should be –6, but he believes the student should also replace $1 - x$ with $-1x$. Prabhdeep forgets to add the constant term of 1 in the first quantity of $(2x + 1)$ and introduces a new error. Only Briannon simplifies the expression correctly by equating the subtracting of quantities and the distributing of a negative.

Activity 4: Exams and Take-Home Projects

"TO ENSURE DEEP, HIGH-QUALITY LEARNING FOR all students, assessment and instruction must be integrated" (NCTM 2000, p. 23). It is essential that

Explain how to fix this simplification. Give reasons.
$(2x + 1) - (x + 6) = 2x + 1 - x + 6$

If x=3 then the order of operations would take place, so the problem would look like (2·3+1) − (3+6) = 2·3+1−3+6 you would have to do 1− 3 instead of 1+6. But its actually 3+6. so that's the mistake.

Gabrielle's solution

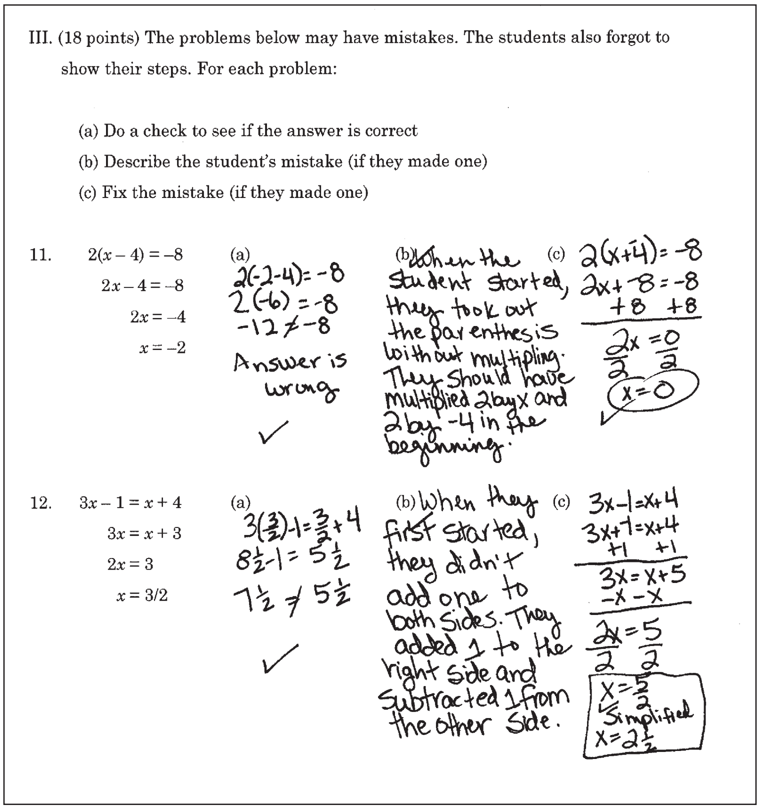
The problem will look like this in its correct form (2x+1) − (x+6) = 2x + ⁻1x + ⁻6 because there is a minus sign right outside of the () On the left side it means its −1. So if you times −1 by x its −1x not 1−x. When you times −1 by 6 its ⁻6 not 6.

Prabdheep's solution

Explain how to fix this problem. Give Reasons
(2x+1) − (x+6) = 2x+1−x(+)6
you are subtracting x and 6 not subtracting x and adding 6
Correctly simplified the problem is
(2x+1)+−(x+6) ← distribute negative
2x+1+−x+−6
x+−5

Briannon's solution

Fig. 3 A sample journal problem, listed at top, is explained three ways by three students.



III. (18 points) The problems below may have mistakes. The students also forgot to show their steps. For each problem:

(a) Do a check to see if the answer is correct
(b) Describe the student's mistake (if they made one)
(c) Fix the mistake (if they made one)

11. $2(x-4)=-8$, $2x-4=-8$, $2x=-4$, $x=-2$

(a) $2(-2-4)=-8$, $2(-6)=-8$, $-12\neq-8$. Answer is wrong ✓

(b) When the student started, they took out the parenthesis without multipling. They should have multiplied 2 by x and 2 by -4 in the beginning.

(c) $2(x+4)=-8$, $2x+8=-8$, $+8\ +8$, $\frac{2x}{2}=\frac{0}{2}$, $x=0$ ✓

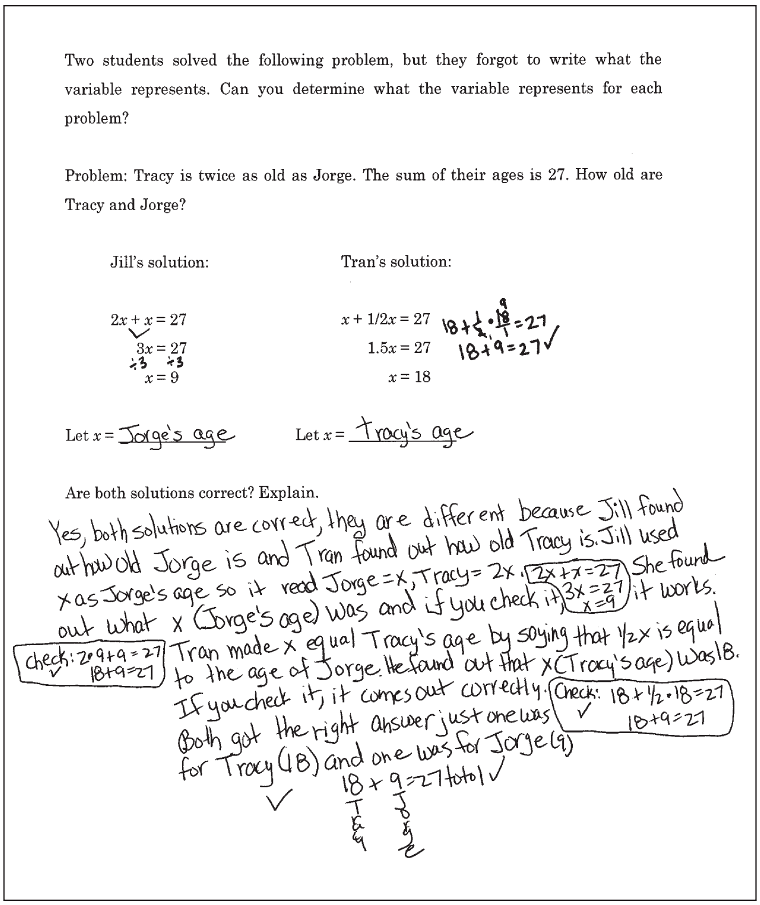
12. $3x-1=x+4$, $3x=x+3$, $2x=3$, $x=3/2$

(a) $3(\frac{3}{2})-1=\frac{3}{2}+4$, $8\frac{1}{2}-1=5\frac{1}{2}$, $7\frac{1}{2}\neq5\frac{1}{2}$ ✓

(b) When they first started, they didn't add one to both sides. They added 1 to the right side and subtracted 1 from the other side.

(c) $3x-1=x+4$, $3x+1=x+4$, $+1\ \ +1$, $3x=x+5$, $-x\ -x$, $\frac{2x}{2}=\frac{5}{2}$, $x=\frac{5}{2}$, Simplified $x=2\frac{1}{2}$

Fig. 4 Sample exam questions



Two students solved the following problem, but they forgot to write what the variable represents. Can you determine what the variable represents for each problem?

Problem: Tracy is twice as old as Jorge. The sum of their ages is 27. How old are Tracy and Jorge?

Jill's solution: $2x+x=27$, $3x=27$, $\div3\ \div3$, $x=9$. Let $x=$ Jorge's age

Tran's solution: $x+1/2x=27$, $1.5x=27$, $x=18$. $18+\frac{1}{2}\cdot\frac{18}{1}=27$, $18+9=27$ ✓. Let $x=$ Tracy's age

Are both solutions correct? Explain.

Yes, both solutions are correct, they are different because Jill found out how old Jorge is and Tran found out how old Tracy is. Jill used x as Jorge's age so it read Jorge = x, Tracy = 2x. She found out what x (Jorge's age) was and if you check it, ($2x+x=27$, $3x=27$, $x=9$) it works. (Check: $2\cdot9+9=27$, $18+9=27$ ✓) Tran made x equal Tracy's age by saying that 1/2x is equal to the age of Jorge. He found out that x (Tracy's age) was 18. If you check it, it comes out correctly. (Check: $18+1/2\cdot18=27$, $18+9=27$ ✓) Both got the right answer just one was for Tracy (18) and one was for Jorge (9). $18+9=27$ total ✓ (T age, J age) ✓

Fig. 5 Another example of an exam question

students have the opportunity to demonstrate their ability for error analysis on tests and projects. This format provides an effective measure of understanding, because students must explain their thinking. Test questions should be short and focus on a single concept. The sample questions in **figure 4** give students the opportunity to demonstrate their understanding of the distributive property in the first question and the addition property of quality on the second problem (see **fig. 4**). Note that there are student errors in problem 12; the simple error is easily found using this format. A third problem assesses students' ability to set up word problems (see **fig. 5**).

Including error analysis as part of a take-home test or project presents students with the opportunity to demonstrate their knowledge on more complicated multistep problems. A typical take-home assessment requires students to identify two or three errors in a single solution, write a correct solution, and justify their reasoning for each step. At home, students do not work under the same time pressure as in-class tests and have the opportunity to express their mathematical thinking more thoughtfully.

Conclusion

ALLOWING OPPORTUNITIES TO DISCUSS AND EXplain common errors can help students become more accurate at solving linear equations and working cooperatively to correct their homework. Students are most effective when they have the opportunity to explain their thinking daily in classroom discussions as well as in smaller collaborative groups. Activities like "pass the pen" and "find an expert" should be used once or twice during a unit to foster active student discussion and participation. As students become better at error analysis, they should be given the opportunity to work alone and explain their ideas in journal writings several times a week and on graded tests and take-home projects at the end of the unit. Questions requiring error analysis are effective tools to assess students' understanding of linear equation solving because error analysis requires a deeper understanding of equation solving than simply "doing the problem."

Bibliography

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