

# Inquiry-Discourse

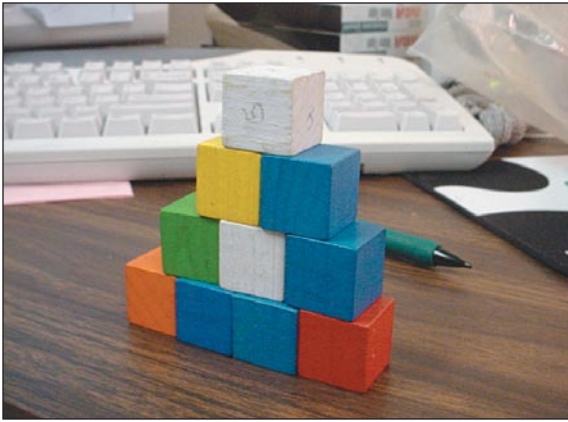
## Mathematics Instruction

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**P** *principles and Standards for School Mathematics* (NCTM 2000) proposes that mathematics instruction provide opportunities for students to engage in mathematical inquiry and in meaning-making through discourse. Mathematics teachers are encouraged to build on student discoveries in designing subsequent instruction. Natural consequences of using an inquiry-based approach to teaching include the emergence of unexpected mathematical results and the articulation of novel and different strategies by students. Anticipating the potential for such occurrences, *Professional Standards for Teaching Mathematics* (NCTM 1991) urges all teachers to remain flexible and responsive to student ideas in their instruction: Help students make connections among various solutions, tie student ideas to important mathematical structures, and extend student inquiry by posing questions and tasks that challenge their initial interpretations of problems or their false generalizations.

In this article, we draw from our efforts in implementing inquiry- and discussion-based mathematics instruction in one classroom to illustrate how students' unexpected interpretations of one problem created an opportunity to extend their mathematical investigations. We also share the particular demands that building on student ideas placed on our curriculum and instruction. In this class, in accordance with the spirit of *Principles and Standards for School Mathematics* (NCTM 2000), students were encouraged to work on problem-solving tasks, discuss their findings with peers, and form and defend conjectures. The sequence of events we describe occurred during the second week of school and in the context of a unit on problem solving with



**Fig. 1** Cereal box display



**Fig. 2** Group 1's interpretation of the solution

which we began the academic year. During this unit, we presented students with several problems each day. They worked on these problems either in small groups during class time or individually as a homework assignment. They then presented their solutions to the larger group. During the large-group discussions, students were asked to verify ideas presented by various individuals and reach consensus on plausible solutions in collaboration with their peers. Through this approach, we hoped to foster not only their problem-solving skills but also the creation of a learning community among them. This style of learning and doing mathematics was completely new to all our students.

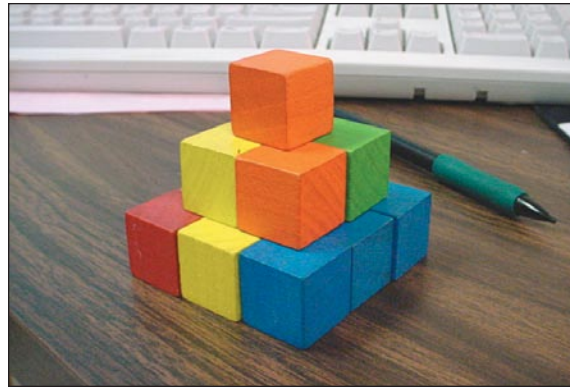
### THE CEREAL BOX PROBLEM

The problem that became the focus of our work, the Cereal Box problem (Masingila, Lester, and Raymond 2002), is as follows:

*Part 1:* A store manager told a sales clerk that 45 cereal boxes had to be stacked in a display window and that all the boxes had to be used. The manager also told the clerk that all the boxes had to be set up in a triangle, as shown in **figure 1**. The sales clerk wondered how many boxes would have to be placed on the bottom row to build a triangle that would use all the boxes.

*Part 2:* What if the clerk had to use 200 boxes in the display? How many boxes would have to be placed on the bottom row to build the triangle?

Our goal in posing this problem was for students to use the strategies of setting up a table, looking for a pattern, and generalizing numerical results. We had hoped that student exploration of this problem would give us an opportunity to discuss Gauss's method for finding the sum of consecutive positive integers and also to introduce triangular numbers. We had planned to use the results of this problem later in the year when we introduced figurate numbers to the class. What occurred in class forced a complete change in our instructional plan.



**Fig. 3** Group 3's interpretation of the solution

### Day 1: Multiple Interpretations

Once students began presenting their solutions, we were surprised to see that three different interpretations of the problem had emerged, each leading to a different answer. The following is a summary of each:

*Group 1:* This group had interpreted the problem to mean that the structure consisted of stacks of 6 cereal boxes, one behind the other (6, 6, ...) (see **fig. 2**) and concluded that, for part 1, the last row of the structure could contain only 3 cereal boxes. The students' answer for part 2 was that the display would consist of 33 rows of 6 cereal boxes each, with 2 boxes left over.

*Group 2:* This group's interpretation was consistent with the model presented in Masingila, Lester, and Raymond (2002). The students assumed that the number of cereal boxes in each row increased by 1 each time (1, 2, 3, ...). For part 1, they suggested that the clerk must place 10 cereal boxes in the last row. For part 2, they suggested that the clerk must form two rows of 20 at the base of the display in order to use up all 200 cereal boxes.

*Group 3:* This group had assumed that at each row a flat square surface was added to the display. The dimension of this square surface increased by one each time (1, 4, 9, ...). (See **fig. 3**). The students concluded that the clerk had to make two displays,

**Table 1**

Total Number of Cereal Boxes in Each Display for a Given Row			
Row Number	Total Number of Cereal Boxes in Display 1	Total Number of Cereal Boxes in Display 2	Total Number of Cereal Boxes in Display 3
1	6	1	1
2	12	3	5
3	18	6	14
4	24	10	30
⋮	⋮	⋮	⋮
10	60	45	$100 + 81 + 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1$
⋮	⋮	⋮	⋮
$n$	$6n$	$n + (n-1) + (n-2) + \dots + 1$	$n^2 + (n-1)^2 + \dots + 1^2$

one with a base consisting of 16 cereal boxes and a smaller one with a base consisting of 9 boxes. There would be 1 box left over. For part 2, the group suggested that the clerk could construct displays in three different sizes: large, with a base consisting of 49 cereal boxes; medium, having 25 cereal boxes as its base; and small, consisting of only two layers, containing 1 and 4 boxes respectively.

None of the students seemed concerned that their solutions did not meet the initial conditions of the problem: building only one display and using all the boxes. In addition, all students had relied on successive addition of row entries to answer each part of the problem. None of them recognized the need to find a formula for determining the total number of cereal boxes in each of the displays. We decided to address both issues. First, we emphasized the need to take into account the given conditions when solving problems. We then asked students to attempt to decide which of the three displays the clerk would have to build in order to use up a total of 1891 cereal boxes. The reason for using a number as large as 1891 was to push students toward generalizing results. Moreover, by including students' own displays in the problem statement, we hoped to increase their interest and investment in the task.

The students constructed a table of values (see **table 1**) in which they recorded the total number of boxes on each given row of each display. Notice that although they offered a recursive formula for determining the total number of cereal boxes used in displays 2 and 3, they could not express these formulas in closed form.

The students quickly observed that the total number of cereal boxes in display 1 was always a multiple of 6. They explained that to determine

whether it was possible to have a certain number of cereal boxes in display 1, they would need to check only for divisibility of the number by 6. Jeremy formalized this observation by proposing that if  $m$  cereal boxes were used in the first row of his group's display, then the total number of cereal boxes would equal  $mn$ , with  $n$  representing the number of rows in the structure. He further elaborated that to determine whether building such a display with a specific number of cereal boxes was possible, one would need to test only whether the number were divisible by both  $m$  and  $n$ . (Divisibility by one or the other is sufficient.)

In examining the second group's display, several students believed that because the rows contained consecutive positive integers, it would be possible to reach a row within the structure at which the sum of cereal boxes would equal 1891 or any positive integer, obviously a false generalization. This assumption was rejected by other students who immediately offered counterexamples from the table of values. Despite these discussions, the students were not yet certain whether it was possible to reach a sum of 1891 cereal boxes in each of the second and third structures. They asked to work on the problem as a homework assignment.

### Day 2: Building Connections

At the beginning of the second day, Ryan reported that he had found a formula for generating the sum of consecutive positive integers. He explained that by multiplying the row number ( $n$ ) by  $(0.5n + 0.5)$  he could predict the total number of cereal boxes at each row of display 2. Without any prompting from us, all the students tested Ryan's formula and were excited to see that it worked. Alayna immediately noticed that by using this formula the clerk could definitely use the second structure as a model to

display 1891 cereal boxes, because 61 satisfied the equation. [She had set the equation  $(n)(0.5n + 0.5) = 1891$  and solved for  $n$ , using both a guess-and-check strategy and a calculator.] She also suggested that we rewrite Ryan's equation, using  $1/2$  instead of  $0.5$  to ease the numerical calculations. Hence, the following generalization was noted:

Ryan's formula:

$$1 + 2 + \dots + n = n \left( \frac{1}{2} \cdot n + \frac{1}{2} \right) = \frac{1}{2}n(n + 1)$$

Although the students were pleased with the progress they had made on the problem, they could not yet tell whether structure 3 would also be a suitable structure for displaying 1891 cereal boxes. As we were deciding whether to "tell" the students the formula for determining the total number of boxes in structure 3, four students volunteered that, in light of Ryan's formula, they had other findings they wanted to share with their peers. Surprisingly, all four had considered extensions of the problem we had assigned.

1. Bridgette said that she had designed a new display that began with a row of 2 cereal boxes in the top row, followed by consecutive even numbers of cereal boxes in each subsequent row (2, 4, 6, 8, ...). She had recognized that the sum of the cereal boxes in her structure was twice that in the second structure, or  $2(1 + 2 + 3 + \dots)$ . Using Ryan's formula, she offered the following generalization for finding the sum of consecutive positive even integers.

Bridgette's formula:

$$2 + 4 + \dots + 2n = 2(1 + 2 + \dots + n) = 2 \left( \frac{1}{2} \right) n(n + 1) = n(n + 1)$$

2. Kerri, too, had created a new display in which cartons containing a cubic number of cereal boxes were placed at each row (1, 8, 27, ...). Her argument was that in crowded warehouses, because of limited space, this stacking process was more feasible. She concluded that in her display the sum of cereal boxes on any given row was the same as its corresponding entry in the second display. Using Ryan's formula, then, she offered the following generalization for finding the sum of cubes of consecutive positive integers.

Kerri's formula:

$$1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n + 1)}{2} \right]^2$$

3. Jamie said that because she could not make any progress on the problem we had assigned, she had considered a display in which rows contained consecutive odd number of cereal boxes (1, 3, 5, ...). She had been intrigued that the total number of cereal boxes on any given row was the same as the row number squared. She presented the following formula for finding the sum of consecutive positive odd integers to the group.

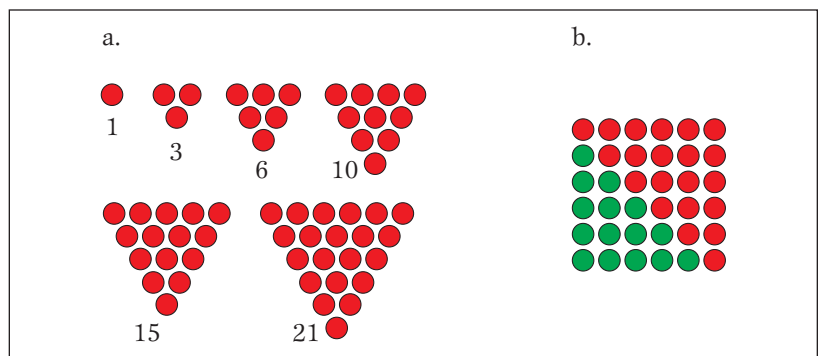
Jamie's formula:

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

4. Erica had chosen to construct a display in which rows contained consecutive triangular numbers of cereal boxes (1, 3, 6, 10, ...). She explained that because she had become obsessed by values in the second display, she decided to construct a display that used the same numbers. Although she could not express the sum of this string of numbers in closed form, she had noticed that the sum of any two consecutive entries in her structure was a perfect square ( $1 + 3 = 4$ ,  $3 + 5 = 9$ , ...). We decided to build on Erica's observation to help students attach a geometric meaning to the various classes of numbers they were exploring. Therefore, we provided the students with a geometric illustration of Erica's conjecture (see **fig. 4**) and mentioned that the entries of the second display were called triangular numbers.

### Day 3: Combining Algebra and Geometry

Indeed, the decision to share a geometric representation of Erica's finding proved to be effective. At the beginning of the fourth day of instruction, Kelly and Ashley asked if they could share their respective groups' geometrically based justifications for Ryan's and Jamie's formulas (see **figs. 5** and **6**). Both groups had used the concept of area to generalize results relative to the sum of consecutive positive integers and consecutive odd numbers. Naomi presented her group's algebraic approach for proving



**Fig. 4** The sum of two consecutive triangular numbers is a square number ( $T_5 + T_6 = 6^2$ ).

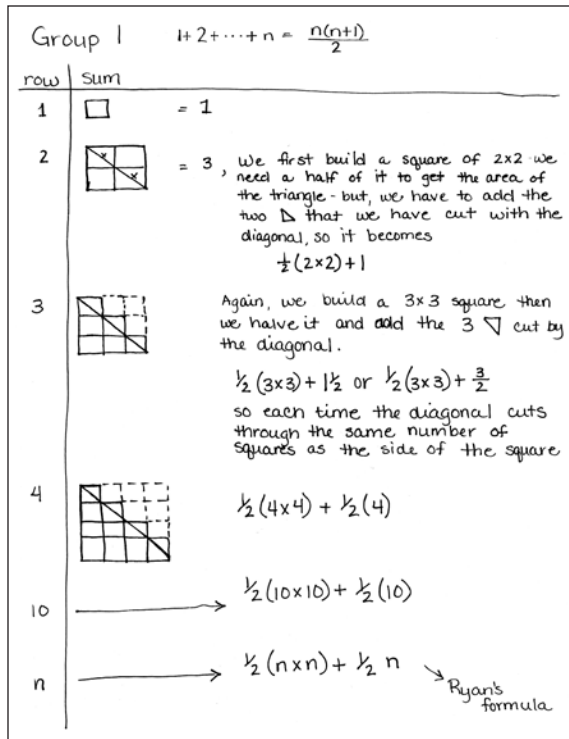


Fig. 5 Kelly's group justified Ryan's formula geometrically.

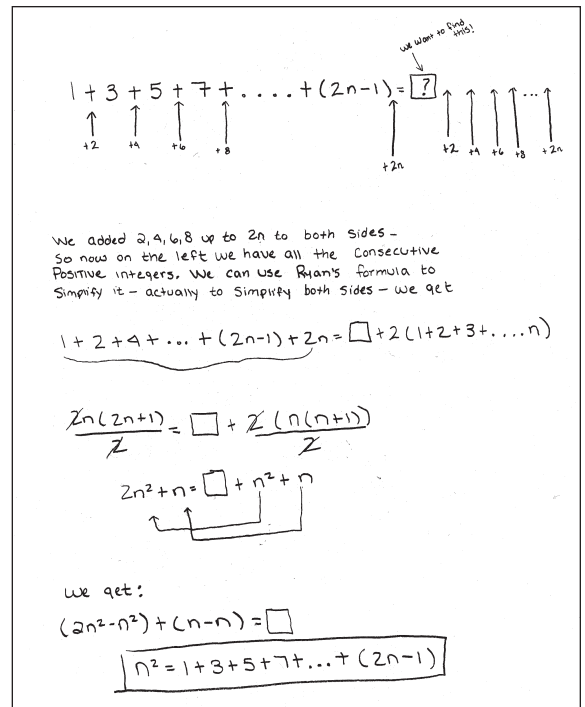


Fig. 7 Naomi's group justified Jamie's conjecture algebraically.

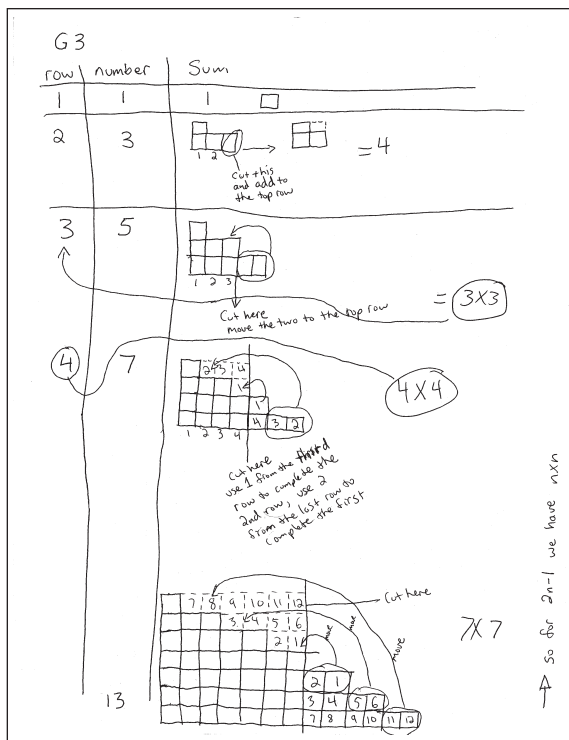


Fig. 6 Ashley's group justified Jamie's formula geometrically.

Jamie's conjecture and explained that her group's proof was closely linked to Ashley's model because it explained the regrouping of the cubes in the latter's geometric model to form squares (see fig. 7).

Bridgette also volunteered a second model for illustrating her own formula for finding the sum of

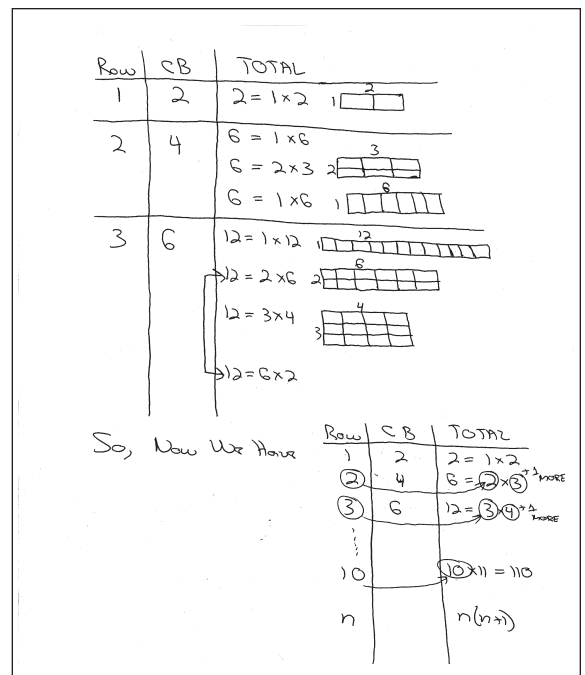


Fig. 8 Bridgette's model justified her own conjecture.

consecutive even integers (see fig. 8). She had first tried building rectangles with areas equal to 2, 6, and 12 (the sum of consecutive positive even numbers for the first three levels). She had noticed that in the first element of the sequence, 2, she could build only one rectangle, which had the dimension 1 · 2. With 6 boxes, she constructed two rectangles, with the dimensions 1 · 6 and 2 · 3. With 12, she constructed three rectangles, with the dimensions

**Table 2**

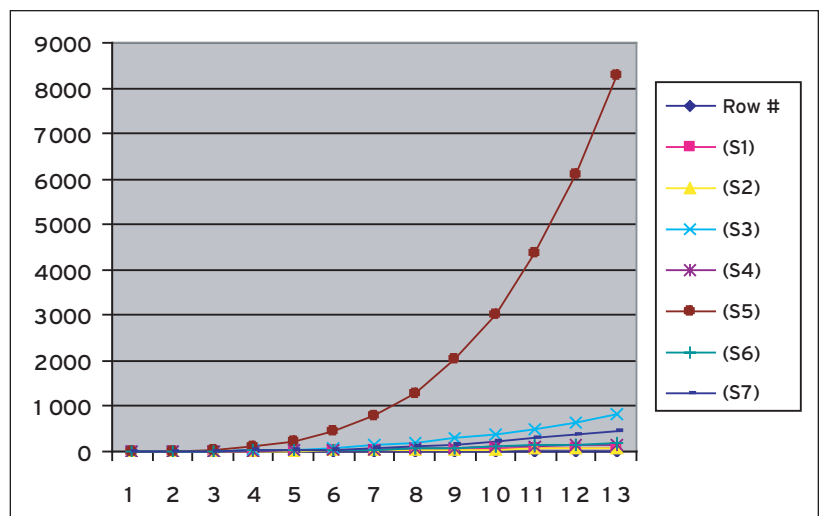
**Cumulative Number of Cereal Boxes in Each Display**

Row #	Display 1 (S1)	Display 2 (S2)	Display 3 (S3)	Jamie's Display (S4)	Kerri's Display (S5)	Bridgette's Display (S6)	Erica's Display (S7)
1	6	1	1	1	1	2	1
2	12	3	5	4	9	6	4
3	18	6	14	9	36	12	10
4	24	10	30	16	100	20	20
5	30	15	55	25	225	30	35
6	36	21	91	36	441	42	56
7	42	28	140	49	784	56	84
8	48	36	204	64	1296	72	120
9	54	45	285	81	2025	90	165
10	60	55	385	100	3025	110	220
11	66	66	506	121	4356	132	286
12	72	78	650	144	6084	156	364
13	78	91	819	169	8281	182	455
14	84	105	1015	196	11025	210	560
15	90	120	1240	225	14400	240	680
16	96	136	1496	256	18496	272	816
17	102	153	1785	289	23409	306	969
18	108	171	2109	324	29241	342	1140
19	114	190	2470	361	36100	380	1330

1 · 12, 2 · 6, and 3 · 4. Using these three special cases, she had observed that the sum of consecutive even numbers was the same as building a rectangle in which one dimension was equal to the order of the number in the set of positive even numbers and a second dimension was equal to one more than the first. This was yet another promising development.

As our students were recording these new conjectures and models, we asked if they had made additional observations that they wanted to share. Ashley pointed out that she had noticed that the total number of cereal boxes in Kelly's display "grew much faster" than others. Amanda recommended that perhaps we could group different displays in "families" according to the rate at which they grew.

We decided to pursue Ashley's and Amanda's comments about the rate of growth of total number of cereal boxes because we were concerned that the focus on numerical and algebraic representation of number structures prevented the students from realizing the amount of useful information they could extract from graphs when solving problems. In doing so, we used electronic spreadsheets, which we saw as an ideal tool to explore the problem further.



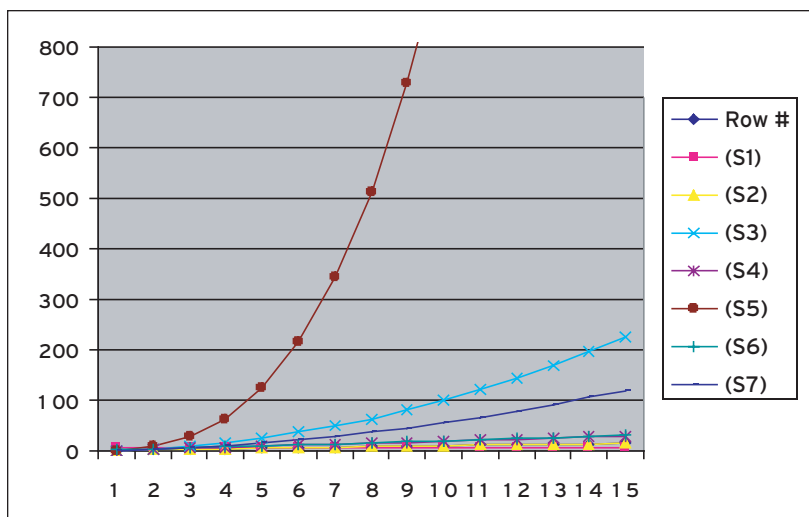
**Fig. 9** Cumulative number of cereal boxes in each display

**Day 4: Extending the Analysis and Studying Graphs**

We began the fifth day by asking students to enter all the data in a spreadsheet and graph the rate of growth of cereal boxes per row as well as the cumulative number of cereal boxes per row for each of the now seven structures (the original three structures as well as the displays of Jamie, Kerri,

**Table 3**

Number of Cereal Boxes in Each Row of Each Display							
Row #	Display 1 (S1)	Display 2 (S2)	Display 3 (S3)	Jamie's Display (S4)	Kerri's Display (S5)	Bridgette's Display (S6)	Erica's Display (S7)
1	6	1	1	1	1	2	1
2	6	2	4	3	8	4	3
3	6	3	9	5	27	6	6
4	6	4	16	7	64	8	10
5	6	5	25	9	125	10	15
6	6	6	36	11	216	12	21
7	6	7	49	13	343	14	28
8	6	8	64	15	512	16	36
9	6	9	81	17	729	18	45
10	6	10	100	19	1000	20	55
11	6	11	121	21	1331	22	66
12	6	12	144	23	1728	24	78
13	6	13	169	25	2197	26	91
14	6	14	196	27	2744	28	105
15	6	15	225	29	3375	30	120
16	6	16	256	31	4096	32	136
17	6	17	289	33	4913	34	153
18	6	18	324	35	5832	36	171
19	6	19	361	37	6859	38	190
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n$	$6(n)$	$n$	$n^*n$	$2n - 1$	$n^*n^*n$	$2n$	$\frac{n(n + 1)}{2}$



**Fig. 10** Number of cereal boxes in each row of each display

Bridgette, and Erica). The students constructed three tables and their accompanying graphs: **table 2** and **figure 9** represented the cumulative number of cereal boxes per row in each structure; **table 3** and **figure 10** represented the total number of

cereal boxes per row in each of the structures; and **table 4** and **figure 11** represented the change in the number of cereal boxes per row.

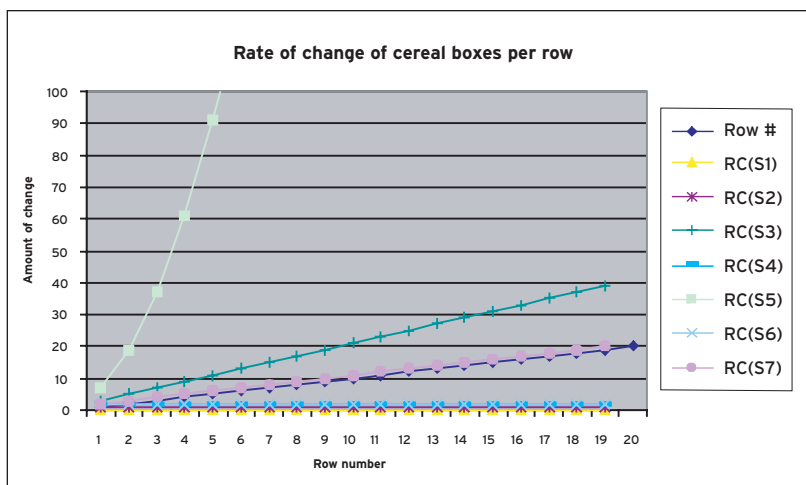
In light of their graphs, some students argued that because the graph of the total number of cereal boxes per row in structure 3 fell between the graphs of structure 7 and structure 5, then the formula for generating the total number of cereal boxes in structure 3, in all likelihood, would be cubic. They supported their conjecture by comparing the corresponding numerical values in each of the three columns. The students also observed that when the graph of the rate of growth of cereal boxes was linear, its corresponding graph, representing the total number of cereal boxes in the same display, was quadratic. They noticed also that when the graph of rate of growth was a horizontal line, the graph of the total number of cereal boxes was linear. In light of these observations, they deduced that the degree of power function that generated the total number of cereal boxes in a structure was one more than the degree of the function that described the rate of change of cereal boxes per row. They speculated, then, that because the graph of the rate of growth of

**Table 4**

Rate of Change of Cereal Boxes per Row in Each Display														
Row #	S1	RC(S1)	S2	RC(S2)	S3	RC(S3)	S4	RC(S4)	S5	RC(S5)	S6	RC(S6)	S7	RC(S7)
1	6	0	1	1	1	3	1	2	1	7	2	2	1	2
2	6	0	2	1	4	5	3	2	8	19	4	2	3	3
3	6	0	3	1	9	7	5	2	27	37	6	2	6	4
4	6	0	4	1	16	9	7	2	64	61	8	2	10	5
5	6	0	5	1	25	11	9	2	125	91	10	2	15	6
6	6	0	6	1	36	13	11	2	216	127	12	2	21	7
7	6	0	7	1	49	15	13	2	343	169	14	2	28	8
8	6	0	8	1	64	17	15	2	512	217	16	2	36	9
9	6	0	9	1	81	19	17	2	729	271	18	2	45	10
10	6	0	10	1	100	21	19	2	1000	331	20	2	55	11
11	6	0	11	1	121	23	21	2	1331	397	22	2	66	12
12	6	0	12	1	144	25	23	2	1728	469	24	2	78	13
13	6	0	13	1	169	27	25	2	2197	547	26	2	91	14
14	6	0	14	1	196	29	27	2	2744	631	28	2	105	15
15	6	0	15	1	225	31	29	2	3375	721	30	2	120	16
16	6	0	16	1	256	33	31	2	4096	817	32	2	136	17
17	6	0	17	1	289	35	33	2	4913	919	34	2	153	18
18	6	0	18	1	324	37	35	2	5832	1027	36	2	171	19
19	6	0	19	1	361	39	37	2	6859	1141	38	2	190	20
20	6	0	20	1	400	41	39	2	8000	1261	40	2	210	21

entries in structure 3 was parabolic, then the function that generated the total number of cereal boxes in that structure must be cubic.

As the reader recognizes, these observations are closely linked to the concepts of the calculus of change. Although this was an exciting mathematical avenue to pursue, we decided to delay discussing this topic until the students had solved the original problems. Therefore, we asked that students make some conjectures about the possible cubic function that represented the data in structure 3. Kelly said that because the formula for the sum of integers was roughly in terms of  $n^2$  and was divided by 2, she believed the sum of squares would be something like  $n^3/3$ . Students immediately created another column on the spreadsheet in which they recorded values of  $n^3/3$  for different  $n$ . Comparing these values with those in structure 3, they stated that although Kelly’s formula ( $n^3/3$ ) was a close “guess,” it did not exactly fit their data in the third structure. A few minutes later, Corey reported that because the formula for finding the sum of consecutive integers contained the factors  $n$ ,  $(n + 1)$ , and  $1/2$ , the formula for structure 3 might look like  $n \cdot (n + 1) \cdot (n + ?)/3$ . We suggested that students use their spreadsheet files and try to find what was missing from Corey’s formula. Using trial and error, they produced the



**Fig. 11** Rate of change of cereal boxes per row in each structure

following generalization for finding the sum of squares of consecutive positive integers:

Corey’s formula:

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)\left(n + \frac{1}{2}\right)}{3}$$



Using Corey's formula, the students found it easy to reach an answer to the Cereal Box problem and concluded that structure 3 would not serve as a suitable model for displaying 1891 cereal boxes. At this point, students had answered the initial problem, posed days earlier. However, they wanted to find a formula for determining the sum of consecutive triangular numbers as presented in structure 7. Using their spreadsheets and a trial-and-error method similar to the one they had used to investigate Corey's conjecture, they successfully generated the following formula for finding the sum of consecutive triangular numbers:

Erica's formula:

$$1 + 3 + 6 + \dots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Our students were so pleased with their own discoveries that they decided to investigate formulas for the sum of fifth and sixth powers of consecutive natural numbers. We closed the day's session by asking them to consider other extensions of the Cereal Box problem that we could explore as a group and bring to class their initial attempts at solving these extensions.

### Days 5-10: Extended Inquiry

The following is a list of conjectures and problems students brought to class:

*Sarah B.* Is it possible to find a number in the sequence  $1^3 + 3^3 + 5^3 + \dots$  [sum of cubes of odd numbers] so that the result would be an even square number?

*Sarah R.* The sum of the sixth powers of two integers can never be a square number.

*Naomi.* What is the sum of consecutive prime numbers? Aside from the trivial case (1, 3, 5), which sequence of prime numbers leads to a square sum?

*Jamie J.* The sum of cubes of consecutive odds is a triangular number.

*Jamie G.* What is  $\frac{1}{1} + \frac{1}{1+3} + \frac{1}{1+3+5} + \dots$ ?

*Tiffany.* We can write some cubes in the following way:

$$\begin{aligned} 1^3 &= 1 = 1 \\ 2^3 &= 8 = 3 + 5 \\ 3^3 &= 27 = 7 + 9 + 11 \\ 4^3 &= 64 = 13 + 15 + 17 + 19 \end{aligned}$$

Can this pattern be generalized? [This was indeed proposed by the early Greeks as the following: The cube of every positive integer is a sum of consecutive odd integers (Ogilvy and Anderson 1988).]

*Andrea.* Is it ever possible to get a square number as the sum of squares? [The famous "cannonball stacking" problem of Lucas (1875, 1877)]

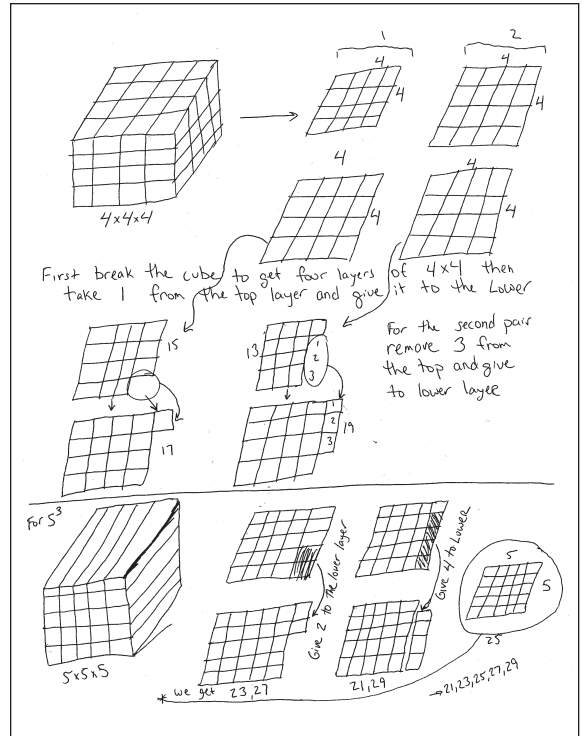


Fig. 12 Paul's visual proof of a conjecture by Tiffany

requires a sum of consecutive squares, beginning with 1, equal to a square. The only nontrivial solution is  $1^2 + 2^2 + 3^2 + \dots + 24^2 = 70^2$ .]

*Amanda.* What is the sum of unit fractions? What is the sum of squares of unit fractions? What is the sum of cubes of unit fractions?

Naturally, we asked that each student select a particular question from the list and investigate it. Indeed, as a result of the students' own insistence, we spent the following five instructional days discussing their findings in class. These investigations led to construction of even more sophisticated mathematical thinking and reasoning on their part. For instance, Paul, after having "played" with Tiffany's question (whether it was possible to write each perfect cube as the sum of a string of consecutive odd numbers) had generated a visual proof to illustrate that it was always possible to do so (see fig. 12).

The following is a brief explanation of his approach.

Here I have two cubes, a  $4 \times 4 \times 4$  and a  $5 \times 5 \times 5$ . I am using two models because the cube can be even or odd, like I have here. If  $n$  is even, as in the case of  $4 \times 4 \times 4$ , then we get an even number of layers of square shape, as here we get 4 layers of 16 [points at his model]. Now, we want consecutive odds, so all we have to do is pair layers. Then remove one from the top layer and add it to the layer right below it. But remember that we want consecutive odds, so if with the first layer I remove 1 block to add to the next layer, with the second pair I remove three blocks and add to its paired layer. This way we can get a sequence of odds.

No matter what number is given, I can always break it into pairs of layers and do this regrouping process.

If  $n$  is odd, as in this case [points to the  $5 \times 5 \times 5$  structure], then when I break it, I have pairs of layers plus one middle level. This middle layer is always odd because  $n$  is odd. I put that aside. Now [when] we work with paired layers again, we must remove 2 from the top layer and add to the one below. [For the] next pair I remove four blocks and add to its paired layers, so on. Again, no matter what  $n$  is, we first set the middle layer, then consider pairs of layers, one above and one below the layer, and continue removing multiples of two blocks from the top layer and adding it to its paired layer below the dividing layer. This way we get the consecutive odds we need. We will do this until we cover all layers.

Building on the line of inquiry that the students had generated themselves, we chose to continue with the study of number theory covering a range of related topics, including proofs without words, recursive sequences, generating functions, and mathematical induction.

## OUR REFLECTIONS ON IMPLEMENTING INQUIRY-DISCOURSE INSTRUCTION

Our goal in this article is to illustrate how unexpected results and even false solutions offered by students can be used to enrich student learning as well as the existing curriculum. We had also hoped to illustrate what NCTM might mean by asking teachers to be “flexible” and “responsive” to student discourse. Notice that although we could have very easily ended the discussion of the Cereal Box problem on the first day by labeling the various solutions that students offered as right or wrong or by giving them the correct formulas, in doing so we would have closed the door not only on their mathematical investigations but also on the formation of a learning community in which members willingly explore mathematics and engage in collaborative construction of knowledge. Being flexible and responsive means that teachers should be deliberate about what topics to pursue with the group as a whole and which to assign as individual work, thoughtful about when to “tell” students and when to let them struggle, and conscious of how to address both short- and long-term curricular goals.

We have also learned that a natural consequence of allowing students to “talk” and “share ideas” in class is the development of mathematical ideas with which we might be unfamiliar. Making sense of students’ informal talk and attaching mathematical meaning to what they say can become difficult at times. In such situations, it might seem feasible to discard student input. We have found that consulting with colleagues and asking them questions about the mathematics that seems ambiguous to us and about the legitimacy of student ideas is quite helpful. These discussions have helped us find ways to

accommodate the mathematical needs of students, push their mathematical thinking forward, and make wise decisions about our next instructional move.

In recent years, there have been some objections to the visions of learning and teaching proposed by the National Council of Teachers of Mathematics. Critics of these visions have suggested that shifting the focus from teacher-directed to student-directed learning and from a hierarchy of skills to simultaneous instruction in all strands of mathematics is a recipe for poor preparation of students for mathematically challenging content (Klein 2006). These critics have further argued that an overemphasis on student input and discovery in learning creates the possibility that there may not be time to cover sufficiently rigorous mathematics in the classroom (Starr 2002; Wu 1997). Our own experiences counter these critics’ assumptions. In the example we describe here, we clearly built on student input and ideas to discuss problems that are usually covered in a number theory course. Given that our students had only a superficial background in elementary mathematics before coming to our class, we consider the quality of their mathematical investigations far more sophisticated than what a traditional curriculum could have hoped to nurture.

It is also reasonable to question whether spending two weeks of instruction on exploring only one problem was a good investment of time. However, we believe that the quality as well as the quantity of mathematics our students explored in the process was substantially more than what we could ever have hoped to cover in class. In the process of our students’ work on the Cereal Box problem, we addressed several important content standards, including patterns and functions, multiple representations, graphing, study of change, algebraic reasoning, and proof. In addition, students made conjectures, formed questions, and collaborated with their peers in constructing mathematical arguments. These skills are far more important than memorizing solutions and answers, which students quickly forget once they leave the classroom.

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